Chiral Skyrmions Research Summary

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Apr 29, 2021

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1 Preliminaries

Given $\vec{m} : \mathbb{R}^2 \to \mathbb{R}^3$ we will write $\nabla \times \vec{m}$ for its curl where we define $\partial_3 \vec{m} := 0$. For $\kappa \in \mathbb{R}$ and $h > 0$ we wish to study the energy functional:

$$E(\vec{m}) = \int_{\mathbb{R}^2} e(\vec{m}) d^2x := \int_{\mathbb{R}^2} \left[\frac{1}{2} |\nabla \vec{m}|^2 + \kappa (\vec{m} - \widehat{e}_3) \cdot (\nabla \times \vec{m}) + \frac{h}{2} |\vec{m} - \widehat{e}_3|^2 \right] d^2x$$

which is defined for $\vec{m} \in \mathcal{M}$ where

$$\mathcal{M} := \{ \vec{m} : \mathbb{R}^2 \to S^2 \ : \ \vec{m} - \widehat{e}_3 \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^3) \}.$$

Notice that if $\vec{m}$ lives in the dense subspace

$$\mathcal{M}_0 := \{ \vec{m} : \mathbb{R}^2 \to \mathbb{R}^3 \ : \ \vec{m} - \widehat{e}_3 \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3) \}$$

then Green's theorem implies that for $R > 0$ such that $\operatorname{supp}(\vec{m} - \widehat{e}_3) \subseteq B_R$ we have

$$\begin{aligned} \int_{\mathbb{R}^2} (\vec{m} - \widehat{e}_3) \cdot (\nabla \times \vec{m}) d^2x &= \int_{\mathbb{R}^2} \vec{m} \cdot (\nabla \times \vec{m}) d^2x - \int_{B_R} (\partial_1 m_2 - \partial_2 m_1) d^2x \\ &= \int_{\mathbb{R}^2} \vec{m} \cdot (\nabla \times \vec{m}) d^2x - \int_{\partial B_R} m_1 dx^1 + m_2 dx^2 \\ &= \int_{\mathbb{R}^2} \vec{m} \cdot (\nabla \times \vec{m}) d^2x \end{aligned}$$

so for $\vec{m} \in \mathcal{M}_0$ we have the following alternate definition of the energy:

$$E(\vec{m}) = \int_{\mathbb{R}^2} \left[\frac{1}{2} |\nabla \vec{m}|^2 + \kappa \vec{m} \cdot (\nabla \times \vec{m}) + \frac{h}{2} |\vec{m} - \widehat{e}_3|^2 \right] d^2x.$$

As an aside, we will henceforth make a certain restriction of κ, h (unless otherwise stated) in order to ensure that $E(\vec{m})$ is non-negative.

Lemma 1.1. *For $h \geq \kappa^2$ we have $E(\vec{m}) \geq 0$ for all $\vec{m} \in \mathcal{M}$. If $h > \kappa^2$ and we let $\delta := 1 - \sqrt{\kappa^2/h}$ then for all $\vec{m} \in \mathcal{M}$ we have*

$$E(\vec{m}) \geq \delta \int_{\mathbb{R}^2} \left[\frac{1}{2}|\nabla\vec{m}|^2 + \frac{h}{2}|\vec{m} - \widehat{e}_3|^2\right] d^2x.$$

As such, we make the following assumption from now on:

We assume $h \geq \kappa^2$.

Notice that $E(\widehat{e}_3) = 0$ and so we seek *topologically non-trivial* minimizers of E. Towards this end, we can compute that

$$\frac{1}{2}|\nabla\vec{m}|^2 = \pm\vec{m} \cdot (\partial_1\vec{m} \times \partial_2\vec{m}) + \frac{1}{2}|\partial_1\vec{m} \pm \vec{m} \times \partial_2\vec{m}|^2$$

and therefore

$$|\vec{m} \cdot (\partial_1\vec{m} \times \partial_2\vec{m})| \leq \frac{1}{2}|\nabla\vec{m}|^2.$$

From the paper we have the following lemma.

Lemma 1.2. *The functional*

$$Q : \mathcal{M}_0 \to \mathbb{R}$$
$$Q(\vec{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \omega(\vec{m}) d^2x := \frac{1}{4\pi}\int_{\mathbb{R}^2} \vec{m} \cdot (\partial_1\vec{m} \times \partial_2\vec{m}) d^2x$$

extends to a continuous map $Q : \mathcal{M} \to \mathbb{R}$ with respect to the strong $W^{1,2}$ topology.

The next result comes from identifying $Q(\vec{m})$ with the Brouwer degree.

Proposition 1.3. *For all $\vec{m} \in \mathcal{M}$, $Q(\vec{m}) \in \mathbb{Z}$.*

Example 1.4. We have $Q(\widehat{e}_3) = 0$.

Example 1.5. Consider, for $\kappa \neq 0$, the map

$$\Phi : \mathbb{R}^2 \to S^2$$
$$\Phi(x) := \left(\operatorname{sign}(\kappa)\frac{2x_2}{1+|x|^2},\ \operatorname{sign}(\kappa)\frac{-2x_1}{1+|x|^2},\ -\frac{1-|x|^2}{1+|x|^2}\right)$$

This map is not in $\mathcal{M}$ since $\Phi - \widehat{e}_3 \notin L^2(\mathbb{R}^2, \mathbb{R}^3)$ (as we'll see later) but nevertheless $Q(\Phi)$ is defined. We compute:

$$\partial_1\Phi(x) = (1+|x|^2)^{-2}\left(-\operatorname{sign}(\kappa)4x_1x_2,\ \operatorname{sign}(\kappa)(4x_1^2 - 2(1+|x|^2)),\ 4x_1\right)$$
$$\partial_2\Phi(x) = (1+|x|^2)^{-2}\left(\operatorname{sign}(\kappa)(2(1+|x|^2) - 4x_2^2),\ \operatorname{sign}(\kappa)4x_1x_2,\ 4x_2\right).$$

For simplicity let's take $\kappa > 0$ for the rest of this computation. Then:

$$\partial_1\Phi \times \partial_2\Phi = (1+|x|^2)^{-3}\begin{pmatrix} -8x_2 \\ 8x_1 \\ 4 - 4|x|^2 \end{pmatrix}$$

so that

$$\Phi \cdot (\partial_1\Phi \times \partial_2\Phi) = \frac{-4}{(1+|x|^2)^2}$$

and therefore

$$Q(\Phi) = -1.$$

Example 1.6. It will be useful to find configurations $\vec{m} : \mathbb{R}^2 \to S^2$ which minimize the chiral interaction density $\kappa\vec{m}\cdot(\nabla\times\vec{m})$ since this is the only term which can give a negative contribution to E. Towards this end, for $x \in B_2 \setminus B_1$ and $\lambda \geq 1$ we set $\psi(\lambda) := \pi - 2\arctan(\lambda)$, $\theta_\lambda(r) := \psi(\lambda)(2-r)$ and define:

$$\vec{m}_\lambda : \overline{B}_2 \setminus B_1 \to S^2$$
$$\vec{m}_\lambda(x) := \left(-\sin(\theta_\lambda(r))\frac{x_2}{|x|},\ \sin(\theta_\lambda(r))\frac{x_1}{|x|},\ \cos(\theta_\lambda(r))\right)$$

Using polar coordinates $x_1 = r\cos(\alpha)$, $x_2 = r\sin(\alpha)$ we can compute that

$$|\nabla\vec{m}_\lambda|^2 = \theta_\lambda'(r)^2 + r^{-2}\sin^2(\theta_\lambda(r)) \quad \text{and} \quad \vec{m}_\lambda\cdot(\nabla\times\vec{m}_\lambda) = \theta_\lambda'(r) + \frac{\sin(\theta_\lambda(r))\cos(\theta_\lambda(r))}{r}.$$

Using the bounds $|\sin(\theta)| \leq |\theta|$ and $1-\cos(\theta) \leq \theta^2/2$ we arrive at

$$\int_{B_2\setminus B_1} e(\vec{m}_\lambda)d^2x \leq C\psi(\lambda)^2 = o(1/\lambda) \quad \text{as } \lambda \to \infty.$$

2 Which non-trivial topological charge yields the smallest energy?

Definition 2.1. For $h \geq \kappa^2$ and $q \in \mathbb{Z}$ we define

$$I_q := \inf\{E(\vec{m}) \ : \ \vec{m} \in \mathcal{M},\ Q(\vec{m}) = q\}.$$

We clearly have $I_0 = 0$. Our first task is to determine for which $q \neq 0$ is I_q the smallest.

Proposition 2.2. *Suppose $h > 0$ and $k \neq 0$. Then $I_{-1} < 4\pi$.*

Proof. For $\lambda \geq 1$ we recall the definition of $\vec{m}_\lambda$ on $B_2 \setminus B_1$ from the previous section and we extend it to all of $\mathbb{R}^2$ via

$$\vec{m}_\lambda(x) := \begin{cases} \Phi(\lambda x) & \text{if } |x| < 1 \\ \vec{m}_\lambda(x) & \text{as defined before if } 1 \leq |x| < 2 \\ \widehat{e}_3 & \text{if } |x| \geq 2. \end{cases}$$

Our earlier computations show that

$$E(\vec{m}_\lambda) = 4\pi\left(1 - \frac{\kappa}{\lambda}\right) + o(1/\lambda) \quad \text{as} \quad \lambda \to \infty.$$

Now, taking $\lambda >> 1$ so that $o(1/\lambda) < 4\pi\kappa/2\lambda$ gives us

$$E(\vec{m}_\lambda) < 4\pi \quad \text{for} \quad \lambda >> 1.$$

An explicit computation then shows $Q(\vec{m}_\lambda) = -1$ as desired. □

Proposition 2.3. *Suppose $h \geq \kappa^2$, $\vec{m} \in \mathcal{M}$ and $Q(\vec{m}) \geq 0$. Then $E(\vec{m}) \geq 4\pi Q(\vec{m})$. In particular:*

$$\textit{if } h > \kappa^2 \textit{ then } I_q \geq 4\pi q \textit{ for } q \in \mathbb{Z}_{\geq 0}.$$

Proof. This follows from the identity

$$\frac{1}{2}|\nabla\vec{m}|^2+\kappa\vec{m}\cdot(\nabla\times\vec{m})=\frac{1}{2}\left(|\partial_1\vec{m}-\kappa\widehat{e}_1\times\vec{m}|^2+|\partial_2\vec{m}-\kappa\widehat{e}_2\times\vec{m}|^2-\kappa^2(1+m_3^2)\right)$$

and our standing assumption that $h\geq\kappa^2$. $\square$

Proposition 2.4. *Suppose $h\geq 2\kappa^2$ and $\vec{m}\in\mathcal{M}$. Then $E(\vec{m})\geq 2\pi|Q(\vec{m})|$. In particular:*

$$\textit{if } h\geq 2\kappa^2 \textit{ then } I_q\geq 2\pi|q|.$$

Proof. This follows from the application of Young's inequality which yields

$$\Big|\int_{\mathbb{R}^2}\kappa(\vec{m}-\widehat{e}_3)\cdot(\nabla\times\vec{m})d^2x\Big|\leq\frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\kappa^2}{h}|\nabla\vec{m}|^2+h|\vec{m}-\widehat{e}_3|^2\right]d^2x$$

and the identity we wrote down right before Lemma 1.2. $\square$

Corollary 2.5. *For $h>\kappa^2$ we have $I_{-1}<I_q$ for all $q\in\mathbb{Z}_{\geq 1}$. If $h\geq 2\kappa^2$ then $I_{-1}<I_q$ for all $q\in\mathbb{Z}\setminus\{0,-1\}$.*

3 Finding a minimizer

Motivated by the results of the previous section we seek a $\vec{m}\in\mathcal{M}$ such that:

$$Q(\vec{m})=-1 \quad\text{and}\quad E(\vec{m})=I_{-1}.$$

The results of the first two sections allow us to obtain a decent candidate for this $\vec{m}$.

Proposition 3.1. *Suppose $h>\kappa^2$. Then there exists a sequence $\vec{m}_k\in\mathcal{M}$ and $\vec{m}_\infty\in\mathcal{M}$ such that $Q(\vec{m}_k)=-1$ for all $k\in\mathbb{Z}_{\geq 0}$ and:*

1. *$\lim_{k\to\infty}E(\vec{m}_k)=I_{-1}$,*
2. *as $k\to\infty$, $\vec{m}_k-\widehat{e}_3$ converges to $\vec{m}_\infty-\widehat{e}_3$ weakly in $W^{1,2}(\mathbb{R}^2,\mathbb{R}^3)$ as well as pointwise almost everywhere.*

Notice now that $h>\kappa^2$ allows us to use Fatou's lemma which in turn implies that

$$E(\vec{m}_\infty)\leq\liminf_{k\to\infty}E(\vec{m}_k)=I_{-1}.$$

So if we can show that $Q(\vec{m}_\infty)=-1$ it will follow that

$$E(\vec{m}_\infty)=I_{-1}\ \text{ as desired.}$$

However $Q(\vec{m}_\infty)=-1$ is not immediate since $Q:\mathcal{M}\to\mathbb{Z}$ is a priori only continuous in the strong $W^{1,2}$ topology, whereas our convergence is in the weak $W^{1,2}$ topology. The paper remedies this with a *concentration-compactness* argument.
Let's denote

$$\rho_k:=|\nabla\vec{m}_k|^2+|\vec{m}-\widehat{e}_3|^2.$$

Definition 3.2. We say that a sequence f_k of non-negative L^1 functions on $\mathbb{R}^2$ is **uniformly tight** if and only if for every $\epsilon > 0$ there exists an $R > 0$ such that

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^2\setminus B_R} f_k(x)d^2x < \epsilon.$$

The key conclusion of the concentration-compactness argument is that, after potentially translating each $\vec{m}_k$ in our sequence, we may assume that the sequence ρ_k is uniformly tight.

Theorem 3.3. First Concentration-Compactness Lemma
Let f_k be a sequence of non-negative functions in $L^1(\mathbb{R}^n)$ with $\|f_k\|_{L^1(\mathbb{R}^n)}$ uniformly bounded above and away from zero. Then there exists a subsequence f_{k_j} which satisfies one of the three following possibilities:

1. **(compactness)** *there exists $x_j \in \mathbb{R}^n$ such that the sequence of functions $\widetilde{f}_j(x) := f_{k_j}(x + x_j)$ is uniformly tight,*
2. **(vanishing)** *for every $R > 0$, $\lim_{j\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} f_{k_j}(x)d^nx = 0$,*
3. **(dichotomy)** *there exists a $0 < t < 1$ such that for every $\epsilon > 0$ there are non-negative L^1 functions $g_j^{1,\epsilon}$ and $g_j^{2,\epsilon}$ on $\mathbb{R}^n$ satisfying* $\mathrm{dist}(\mathrm{supp}(g_j^{1,\epsilon}, g_j^{2,\epsilon})) \to \infty$ *as $j \to \infty$,*

$$\left|\int_{\mathbb{R}^n} g_j^{1,\epsilon}(x)d^nx - t\int_{\mathbb{R}^n} f_{k_j}(x)d^nx\right| \le \epsilon \text{ and } \left|\int_{\mathbb{R}^n} g_j^{2,\epsilon}(x)d^nx - (1-t)\int_{\mathbb{R}^n} f_{k_j}(x)d^nx\right| \le \epsilon \text{ for all } j.$$

We will now demonstrate that neither vanishing nor dichotomy can occur for our sequence ρ_k.

Lemma 3.4. *Let $\vec{m}_k \in \mathcal{M}$ be any sequence with $Q(\vec{m}_k) \neq 0$ for all k, $h > \kappa^2$, and $\rho_k := |\nabla\vec{m}_k|^2 + |\vec{m}_k - \widehat{e}_3|^2$ vanishing in the sense of the above theorem. Then*

$$\liminf_{k\to\infty} E(\vec{m_k}) \ge 4\pi.$$

Proof. The idea here is to show that "vanishing" implies that both the $\kappa\vec{m}\cdot(\nabla\times\vec{m})$ and the $|\vec{m} - \widehat{e}_3|^2$ terms of the energy vanish as $k \to \infty$ and therefore I_{-1} is given by the liminf of the $\frac{1}{2}|\nabla\vec{m}|^2$ terms along. However, our inequality from directly before Lemma 1.2 then gives us a contradiction. □

Lemma 3.5. *Given our sequence $\vec{m_k}$ with $E(\vec{m_k}) \to I_{-1}$ and $Q(\vec{m}_k) = -1$, dichotomy does not occur.*

Proof. Basically, if dichotomy occurs then $\vec{m}_k$ splits into a configuration of charge q_k and a configuration of charge $-(1 + q_k)$. Our $I_{-1} < 4\pi$ bound together with our $I_q \ge 4\pi q$ for $q \ge 0$ bounds imply that we must have $q_k \in \{0, -1\}$ for all k. Without loss of generality one can take $q_k = -1$ for all k (after passing to a subsequence) to get $I_{-1} \le tI_{-1}$ for the $0 < t < 1$ from the definition of dichotomy, a contradiction. □

The above lemmas imply that, up to translating our sequence (which doesn't affect energy or topological charge) we may assume ρ_k is uniformly tight.

Proposition 3.6. *The functionals* $F_{\pm}(\vec{m}) := E(\vec{m}) \pm 4\pi Q(\vec{m})$ *are weakly lower semicontinuous on uniformly tight sequences. In particular, if* $\vec{m}_k$ *is our above sequence then*

$$F_{\pm}(\vec{m}_{\infty}) \leq \liminf_{k \to \infty} F_{\pm}(\vec{m}_k).$$

Proof. The idea here is to use uniform tightness to reduce the problem to integrals over some fixed large ball B_R and then apply Rellich compactness and similar results. □

Corollary 3.7. *Given* $\vec{m}_k, \vec{m}_{\infty}$ *as above we have* $Q(\vec{m}_{\infty} = -1$ *and so* $E(\vec{m}_{\infty}) = I_{-1}$.

4 Short Proof Summary

Before summarizing let's state the main theorem of the paper.

Theorem 4.1. *Suppose* $h > \kappa^2$ *and* $\kappa \neq 0$. *Then there exists a* $\vec{m}_* \in \mathcal{M}$ *with* $Q(\vec{m}_*) = -1$ *so that*

$$I_{-1} = E(\vec{m}_*) < 4\pi.$$

Furthermore, we have

$$I_{-1} < I_q \text{ for all } q > 0 \text{ and if } h \geq 2\kappa^2 \text{ then } I_{-1} < I_q \text{ for all } q \notin \{0, -1\}.$$

1. The fact that $I_{-1} < 4\pi$ is proven by explicitly constructing test configurations $\vec{m}_{\lambda}$. These are given by truncating stereographic projection to the constant map at $\widehat{e}_3$ outside of B_2 and interpolating on $B_2 \setminus B_1$ via a configuration that minimizes the chiral interaction.

2. The inequality $I_{-1} < I_q$ for all $q > 0$ comes from proving $I_q \geq 4\pi q$ for all $q > 0$. This is from rewriting $\frac{1}{2}|\nabla\vec{m}|^2 + \kappa\vec{m} \cdot (\nabla\vec{m})$ in terms of the *helical derivatives* $\partial_j\vec{m} - \kappa\widehat{e}_j \times \vec{m}$.

3. The version of the above result for $q < -1$ comes from $I_q \geq 2\pi|q|$ for all q if we assume $h \geq 2\kappa^2$. This comes from bounding the chiral iteraction above and below by the other parts of the energy using Young's inequality.

4. Taking a minimizing sequence of $\vec{m}_k$'s with $Q(\vec{m}_k) = -1$ we can rule out "vanishing" and "dichotomy" in the concentration-compactness lemma using the above inequalities. Thus we may assume our sequence is uniformly tight and therefore apply weak lower semicontinuity results that only hold on balls B_R with finite radius to conclude.